$3.7 DIMENSIONAL ANALYSIS OF THE EQUATIONS OF CHANGE

- Suppose that we have taken experimental data on, or made photographs of, the flow through some system that cannot be analyzed by solving the equations of change analytically.
  - Example: Flow of a fluid through an orifice meter in a pipe.
- How to scale up (or down) in order to build a new one with exactly the same flow patterns?

Cont'd

- Pre-requisites:
  - Geometric similarity: similar ratios of all dimensions of the pipe and orifice plate in the original system and in the scaled-up (or scaled-down) system
  - Dynamic similarity: similar dimensionless groups (e.g. Re) in the differential equations and boundary conditions
- The study of dynamic similarity is best understood by writing the equations of change, along with boundary and initial conditions, in dimensionless form

For simplicity, let's work on constant ρ and μ

$$(\nabla \cdot \mathbf{v}) = 0$$

$$\rho \frac{D}{Dt}\mathbf{v} = -\nabla \mathcal{P} + \mu \nabla^2 \mathbf{v}$$

a. Define scale factors such as characteristic length, characteristic velcity etc.

b. define dimensionless variables and differential operators

$$\breve{x} = \frac{x}{l_0} \qquad \breve{y} = \frac{y}{l_0} \qquad \breve{z} = \frac{z}{l_0} \qquad \breve{t} = \frac{v_0 t}{l_0} \tag{3.7-3}$$

$$\breve{\mathbf{v}} = \frac{\mathbf{v}}{v_0} \qquad \breve{\mathcal{P}} = \frac{\mathcal{P} - \mathcal{P}_0}{\rho v_0^2} \quad \text{or} \quad \breve{\breve{\mathcal{P}}} = \frac{\mathcal{P} - \mathcal{P}_0}{\mu v_0 / l_0} \tag{3.7-4}$$

for high Re (first form) / for low Re (second form)

$$\breve{\nabla} = l_0 \nabla = \boldsymbol{\delta}_x(\partial/\partial\breve{x}) + \boldsymbol{\delta}_y(\partial/\partial\breve{y}) + \boldsymbol{\delta}_z(\partial/\partial\breve{z}) \tag{3.7-5}$$

$$\breve{\nabla}^2 = (\partial^2/\partial\breve{x}^2) + (\partial^2/\partial\breve{y}^2) + (\partial^2/\partial\breve{z}^2) \tag{3.7-6}$$

$$D/D\breve{t} = (l_0/v_0)(D/Dt) \tag{3.7-7}$$

$$(\nabla \cdot \mathbf{v}) = 0$$

$$\rho \frac{D}{Dt}\mathbf{v} = -\nabla\mathcal{P} + \mu\nabla^2\mathbf{v}$$

$$(\breve{\nabla} \cdot \breve{\mathbf{v}}) = 0$$

$$\frac{D}{D\breve{t}}\breve{\mathbf{v}} = -\breve{\nabla}\breve{\mathcal{P}} + \left[\frac{\mu}{l_0 v_0 \rho}\right]\breve{\nabla}^2\breve{\mathbf{v}}$$

OR :

$$\frac{D}{D\breve{t}}\breve{\mathbf{v}} = -\left[\frac{\mu}{l_0 v_0 \rho}\right]\breve{\nabla}\breve{\breve{\mathcal{P}}} + \left[\frac{\mu}{l_0 v_0 \rho}\right]\breve{\nabla}^2\breve{\mathbf{v}}$$

$$\mathrm{Re} = \left[\frac{l_0 v_0 \rho}{\mu}\right]$$

Relative importance of inertial and viscous forces in the fluid system

What happens for low and high Re?

$$(\breve{\nabla} \cdot \breve{\mathbf{v}}) = 0 \tag{3.7-8}$$

Euler equation

$$\frac{D}{D\breve{t}}\breve{\mathbf{v}} = -\breve{\nabla}\breve{\mathcal{P}} + \left[\frac{\mu}{l_0 v_0 \rho}\right]\breve{\nabla}^2\breve{\mathbf{v}} \qquad \mathrm{Re} \to \infty \tag{3.7-9a}$$

or

$$\frac{D}{D\breve{t}}\breve{\mathbf{v}} = -\left[\frac{\mu}{l_0 v_0 \rho}\right]\breve{\nabla}\breve{\breve{\mathcal{P}}} + \left[\frac{\mu}{l_0 v_0 \rho}\right]\breve{\nabla}^2\breve{\mathbf{v}} \qquad \mathrm{Re} << 1 \tag{3.7-9b}$$

Creeping Flow Eqn

Other dimensionless groups may arise in the initial and boundary conditions with fluid-fluid interfaces :

$$\mathrm{Fr} = \left[\frac{v_0^2}{l_0 g}\right] = \textit{Froude number}$$

$$\mathrm{We} = \left[\frac{\sigma}{l_0 v_0^2 \rho}\right] = \textit{Weber number}$$

EX 3.7-1 Transverse Flow around a Circular Cylinder

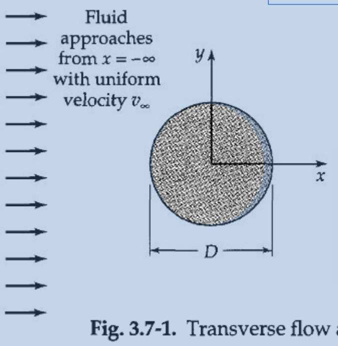
Setting: Steady state experiment. incompressible Newtonian
Find: (1) P and flow patterns as a f(D, L,v_∞, ρ and μ)
(2) Show how to organize the work so that the number of experiments needed will be minimized

Imagine you are in the lab and you want to design the experiment. Do you use Cartesian or Cylindrical System?

Description

1. Consider an idealized flow system: a cylinder of diameter D and length L, submerged in an otherwise unbounded fluid of constant ρ and μ.
2. Initially the fluid and the cylinder are both at rest.
3. At time t = 0, the cylinder is abruptly made to move with v, in the negative x direction.

Solution:



$$(\nabla \cdot \mathbf{v}) = 0$$

$$\rho \frac{D}{Dt}\mathbf{v} = -\nabla \mathcal{P} + \mu \nabla^2 \mathbf{v}$$

I.C.	if $x^2 + y^2 > \frac{1}{4}D^2$ or if $\lvert z \rvert > \frac{1}{2}L$,	$\mathbf{v} = \boldsymbol{\delta}_x v_\infty$
B.C. 1	as $x^2 + y^2 + z^2 \to \infty$,	$\mathbf{v} \to \boldsymbol{\delta}_x v_\infty$
B.C. 2	if $x^2 + y^2 \le \frac{1}{4}D^2$ and $\lvert z \rvert \le \frac{1}{2}L$,	$\mathbf{v} = 0$
B.C. 3	as $x \to -\infty$ at $y = 0$,	$\mathcal{P} \to \mathcal{P}_\infty$

Fig. 3.7-1. Transverse flow around a cylinder.

Dimensionless form of governing eqs with: characteristic length D, velocity v_∞, and modified pressure $\wp_\infty$,

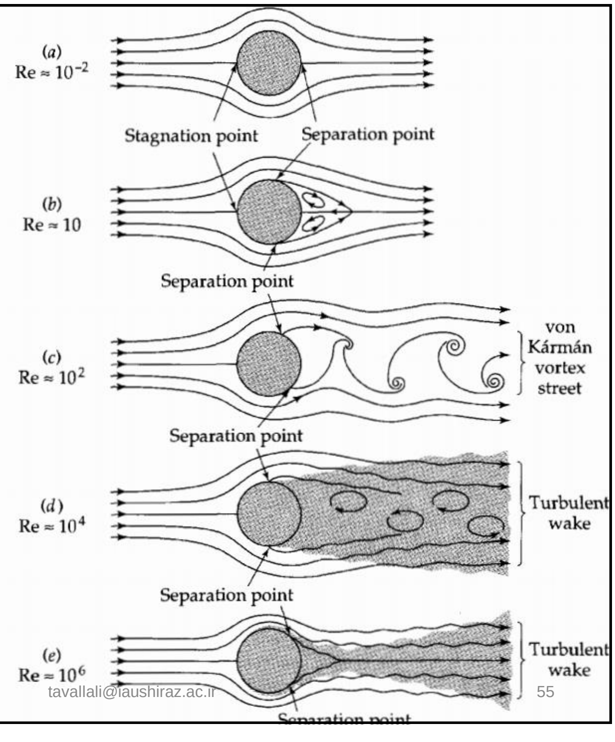
$$(\breve{\nabla} \cdot \breve{\mathbf{v}}) = 0, \quad \text{and} \quad \frac{\partial \breve{\mathbf{v}}}{\partial \breve{t}} + [\breve{\mathbf{v}} \cdot \breve{\nabla} \breve{\mathbf{v}}] = -\breve{\nabla} \breve{\mathcal{P}} + \frac{1}{\text{Re}} \breve{\nabla}^2 \breve{\mathbf{v}} \qquad \text{Re} = D v_\infty \rho / \mu.$$

I.C.	if $\breve{x}^2 + \breve{y}^2 > \frac{1}{4}$ or if $\lvert \breve{z} \rvert > \frac{1}{2}(L/D)$,	$\breve{\mathbf{v}} = \boldsymbol{\delta}_x$
B.C. 1	as $\breve{x}^2 + \breve{y}^2 + \breve{z}^2 \to \infty$,	$\breve{\mathbf{v}} \to \boldsymbol{\delta}_x$
B.C. 2	if $\breve{x}^2 + \breve{y}^2 \le \frac{1}{4}$ and $\lvert \breve{z} \rvert \le \frac{1}{2}(L/D)$,	$\breve{\mathbf{v}} = 0$
B.C. 3	as $\breve{x} \to -\infty$ at $\breve{y} = 0$,	$\breve{\mathcal{P}} \to 0$

Possible solution: $\check{\mathbf{v}} = \check{\mathbf{v}}(\check{x}, \check{y}, \check{z}, \check{t}, \mathrm{Re}, L/D) \quad \text{and} \quad \check{\mathcal{P}} = \check{\mathcal{P}}(\check{x}, \check{y}, \check{z}, \check{t}, \mathrm{Re}, L/D)$

1. We have not solved the flow problem, but have decided on a convenient set of dimensionless variables to suggest the form of the solution.
2. The analysis shows that if we wish to catalog the flow patterns, it will suffice to record them (e.g., photographically) for a series of Re and L/D values; thus, separate investigations into the roles of L, D, v_{∞}, and ρ are unnecessary.
3. Such a simplification saves a lot of time and expense. Similar comments apply to the tabulation of numerical results, in the event that one decides to make a numerical assault on the problem.

Various flow regimes observed as the Re increases



Regions of turbulent flow are shaded in gray.

Scale-up Practice

$$\breve{\mathbf{v}} = \breve{\mathbf{v}}(\breve{x}, \breve{y}, \breve{z}, \breve{t}, \text{Re}, L/D) \quad \text{and} \quad \breve{\mathcal{P}} = \breve{\mathcal{P}}(\breve{x}, \breve{y}, \breve{z}, \breve{t}, \text{Re}, L/D)$$

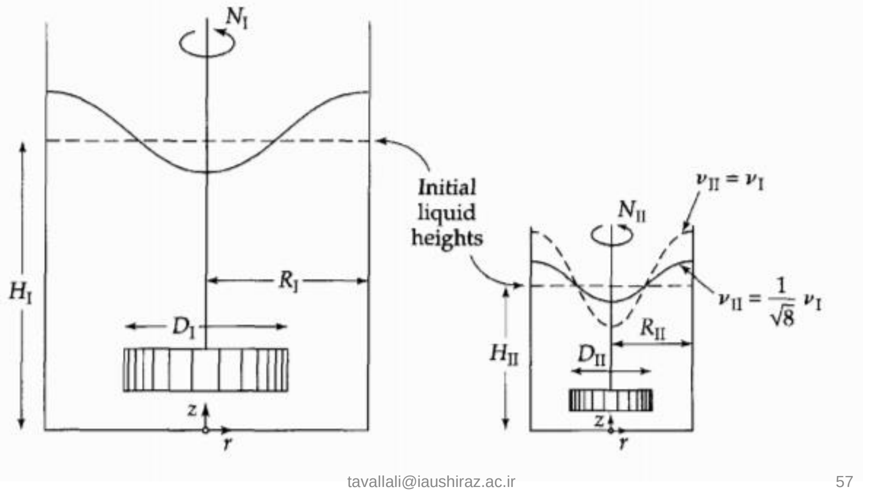
- Predict the flow patterns around a cylinder of $D = 5\ ft$, around which air is to flow with an $v_\infty = 30\ {}^{ft}/_{s}$, by means of an experiment on a scale model of diameter $D_{II} = 1\ ft$.

- Dynamic similarity : $Re_I = Re_{II}$ → if same fluid, then $\frac{\mu_I}{\rho_I} = \frac{\mu_{II}}{\rho_{II}}$, so $v_\infty = 150\ {}^{ft}/_{s}$

EX 3.7-2 Steady Flow in an Agitated Tank

Find: The flow behavior in a large, unbaffled tank of oil as a function of the impeller rotation speed (N RPM)

Given: The impeller shape and its initial position $S_{imp}(r, \theta, z) = 0$.



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Approach: Using model experiments in a smaller, geometrically similar system.

Goal: Determine the conditions necessary for the model studies to provide a direct means of prediction.

Assumption:

(1) Neglect the drag of the atmosphere on the liquid surface

(2) Impeller starts the rotation at $t = 0$.

(3) $S_{int}(r, \theta, z, t) = 0$ provides the gas-liquid interface profile

Solution:

$$(\nabla \cdot \mathbf{v}) = 0$$

$$\rho \frac{D}{Dt}\mathbf{v} = -\nabla \mathcal{P} + \mu \nabla^2 \mathbf{v}$$

I.C. $t = 0$, for $0 \leq r < R$ and $0 < z < H$, $\quad \mathbf{v} = 0$

Location of the impeller after N_t rotations

Boundary conditions:

No-slip & im-permeability conditions

tank bottom	at $z = 0$ and $0 \leq r < R$,	$\mathbf{v} = 0$
tank wall	at $r = R$,	$\mathbf{v} = 0$
impeller surface	at $S_{imp}(r, \theta - 2\pi Nt, z) = 0$,	$\mathbf{v} = 2\pi Nr\boldsymbol{\delta}_\theta$

No mass flow through the gas-liquid interface S_{int}

gas–liquid interface at $S_{int}(r, \theta, z, t) = 0$, $\quad (\mathbf{n} \cdot \mathbf{v}) = 0$

n: local unit normal vector of S_{int}

Force balance on an element of S_{int} (or continuity of the normal component of the momentum flux tensor $\boldsymbol{\pi}$)

gas–liquid interface $\quad \mathbf{n}p + [\mathbf{n} \cdot \boldsymbol{\tau}] = \mathbf{n}p_{atm}$

Non-Dimensionlize:

Characteristic quantities: $v_0 = ND$, $l_0 = D$, $\wp_0 = p_{atm}$, along
Dimensionless coordinates $\breve{r} = r/D$, $\breve{z} = z/D$.

$$(\breve{\nabla} \cdot \breve{\mathbf{v}}) = 0$$

$$\frac{D}{D\breve{t}}\breve{\mathbf{v}} = -\breve{\nabla}\breve{\mathcal{P}} + \left[\frac{D^2 N\rho}{\mu}\right]\breve{\nabla}^2\breve{\mathbf{v}}$$

$$\frac{D}{D\breve{t}}\breve{\mathbf{v}} = -\left[\frac{D^2 N\rho}{\mu}\right]\breve{\nabla}\breve{\mathcal{P}} + \left[\frac{D^2 N\rho}{\mu}\right]\breve{\nabla}^2\breve{\mathbf{v}}$$

$\mathrm{Re} = D^2 N\rho/\mu$.

I.C. at $\breve{t} = 0$, for $\breve{r} = \left[\frac{R}{D}\right]$ and $0 < \breve{z} < \left[\frac{H}{D}\right]$, $\breve{\mathbf{v}} = 0$

Boundary conditions:

tank bottom at $\breve{z} = 0$ and $0 < \breve{r} < \left[\frac{R}{D}\right]$, $\breve{\mathbf{v}} = 0$

tank wall at $\breve{r} = \left[\frac{R}{D}\right]$, $\breve{\mathbf{v}} = 0$

impeller surface at $\breve{S}_{imp}(\breve{r}, \theta - 2\pi\breve{t}, \breve{z}) = 0$, $\breve{\mathbf{v}} = 2\pi\breve{r}\boldsymbol{\delta}_\theta$

gas–liquid interface at $\breve{S}_{int}(\breve{r}, \theta, \breve{z}, \breve{t}) = 0$, $(\mathbf{n} \cdot \breve{\mathbf{v}}) = 0$

and $\mathbf{n}\breve{\mathcal{P}} - \mathbf{n}\left[\frac{g}{DN^2}\right]\breve{z} - \left[\frac{\mu}{D^2 N\rho}\right][\mathbf{n} \cdot \breve{\dot{\gamma}}] = 0$

$\mathbf{n}p + [\mathbf{n} \cdot \boldsymbol{\tau}] = \mathbf{n}p_{atm}$ $\boldsymbol{\tau} = -\mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger) + (\frac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v})\boldsymbol{\delta}$ $\dot{\boldsymbol{\gamma}} = \nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger$

$\breve{\dot{\gamma}} = \partial\breve{v}_j/\partial\breve{x}_i) + (\partial\breve{v}_i/\partial\breve{x}_j)$.

Reminder : $\mathrm{Re} = D^2 N\rho/\mu$ and $\mathrm{Fr} = DN^2/g$.

Possible solution:

$$\breve{\mathbf{v}} = \breve{\mathbf{v}}\left(\breve{r}, \theta, \breve{z}, \breve{t}; \frac{R}{D}, \frac{H}{D}, \mathrm{Re}, \mathrm{Fr}\right) \qquad \breve{S}_{\mathrm{int}} = \breve{S}_{\mathrm{int}}\left(\breve{r}, \theta, \breve{z}, \breve{t}; \frac{R}{D}, \frac{H}{D}, \mathrm{Re}, \mathrm{Fr}\right) = 0$$

$$\breve{\mathcal{P}} = \breve{\mathcal{P}}\left(\breve{r}, \theta, \breve{z}, \breve{t}; \frac{R}{D}, \frac{H}{D}, \mathrm{Re}, \mathrm{Fr}\right)$$

For time-smoothed observations at large t, the dependence on t will disappear, as will the dependence on θ for this axisymmetric tank geometry

Necessary conditions for the proposed model experiment: the two systems must be:

- Geometrically similar (same values of R/D and H/D, same impeller geometry and location),
- Operated at the same values of the Reynolds and Froude numbers. $\frac{D_{\mathrm{I}} N_{\mathrm{I}}^2}{g_{\mathrm{I}}} = \frac{D_{\mathrm{II}} N_{\mathrm{II}}^2}{g_{\mathrm{II}}}$ $\frac{D_{\mathrm{I}}^2 N_{\mathrm{I}}}{\nu_{\mathrm{I}}} = \frac{D_{\mathrm{II}}^2 N_{\mathrm{II}}}{\nu_{\mathrm{II}}}$ $\nu = \mu/\rho$

So: $N_{II}/N_I = (D_I/D_{II})^{1/2}$ $\nu_{II}/\nu_I = (D_{II}/D_I)^{3/2}$

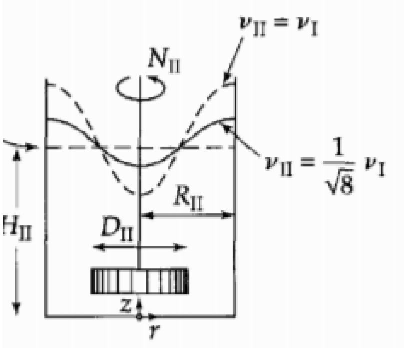
NOTE!

- Important: including the boundary conditions in a dimensional analysis.
- Fr appeared only in the free-surface BC.

$$\mathbf{n}\breve{\mathcal{P}} - \mathbf{n}\left\|\frac{g}{DN^2}\right\|\breve{z} - \left\|\frac{\mu}{D^2 N\rho}\right\|[\mathbf{n} \cdot \breve{\dot{\gamma}}] = 0$$

- Failure to use this condition would result in the omission of the restriction in $N_{II}/N_I = (D_I/D_{II})^{1/2}$

– One might improperly choose $\nu_I = \nu_{II}$. If one did this, with $Re_I = Re_{II}$, the Fr in the smaller tank would be too large, and the vortex would be too deep



HW3

- Solve the following problem at the end of chapter 3 and hand-in the next session: B4, B5, and B7
- For preparing for the mid-term exam, try to check all the problems, specially the followings:
  - A7, B1, B9, B10, B16 and C1

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